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Design Constraints of Branching and Fractal-Like Networks

Ahmad Fakheri[1]

Tony Jabbour, and Mohamed Abdelmajid[2]

Department of Mechanical Engineering

Bradley University, 1501 W Bradley Ave, Peoria, IL 61625, USA

Abstract

The aim of this project is to determine the conditions under which branching systems are a viable alternative. There are abundant examples of branching patterns in the nature prompting a large number of studies trying to explain this naturally occurring phenomenon. A general analytical model for predicting the total pressure drop in a branching system is presented. The predictions of the model are compared with the available results, including the numerical predictions of three dimensional flows, showing the model is reasonably accurate, and can be used to predict the impact of different parameters. The model is then used to develop criteria under which branching systems are advantageous. It is shown that some of the generally believed assumptions about the branched systems may not be valid, including one of the predictions of the Constructal theory.

Nomenclature

A_b	Branching surface area
c	Number of children
d_h	Hydraulic diameter
L	Branching pipe length
$\dot{m}$	Mass flow rate
n	Number of generations
Re	Reynolds number
t	Pipe thickness
ΔP	Pressure Drop
V	Velocity
x	Straight distance between inlet and outlet
λ	Length ratio
δ	Diameter ratio
θ	Half branch Angle
μ	Dynamic viscosity
ν	Kinematic viscosity
ρ	Fluid density
ρ_s	Solid density

[1] Professor, corresponding authorahmad@bradley.edu
[2] Graduate Students

Introduction

Contrary to manmade flow systems, the flow systems in nature rarely involve flow in long straight passages. The common feature amongst air flow through lungs; water flow through river systems; nutrients flow through tree roots and branches; blood flow through arteries, vein and capillaries is that they all have branching pattern. The reason why nature has a preference for branching patterns and their characteristics have been subject of speculation and research for centuries.

The earliest attempt to model this phenomenon appears to have been by Thomas Young in 1808 [1] in his efforts to study flow resistance of an arterial system. He assumed a ratio of 1.26(=$2^{1/3}$):1 for the diameters of parent and children vessels and calculated that twenty-nine bifurcations were necessary to diminish the aorta to the size of the capillaries. Young did not provide a justification for why he chose that ratio. [2].

Murray [3] considered the geometry of branching junctions in mammalian cardiovascular networks and postulated that evolution and natural selection must have resulted in an optimum system that is a compromise between the cost of building and maintaining the network, which is proportional to the square of the radius, and the cost of transporting fluid through the network, which for laminar flow in a cylindrical tube is given by Poiseuille's flow and is inversely proportional to radius to the fourth power [4]. The formulation results in the volumetric flow rate to be proportional to the cube of the vessel diameter

$$\dot{V} = kd^3 \tag{1}$$

Using this and the conservation of mass at a branching point result in the following relation for diameters in successive generations

$$d_i^3 = d_{i+1,1}^3 + d_{i+1,2}^3 + \ldots \tag{2}$$

which for identical children and bifurcating system reduces to

$$d_i^3 = 2d_{i+1}^3 \tag{3}$$

which is identical to what Young had proposed. The Murray's law is based on two biologically reasoned assumptions[2] that evolution and natural selection lead to optimality in (a) geometry and (b) maintenance. The approach has also been extended to nonliving systems, including engineered ones, by including maintenance and operating/energy costs.

Some studies [5-6] have focused on advantages of branching systems under imposed constraints, such as a fixed volume or area. As much it sounds reasonable for that to be the case, there is no known natural law, including evolution and natural selection that requires nature to strive for efficiency, or operate optimally, or to have limits on volume or area. Evolution, has no directional preference for traits, as long as the traits have no effect on procreation or their impact is manifested past that time, those traits will persist and are passed on to the future generations.

In this paper, a general solution for pressure drop in branching systems is derived and used to determine conditions under which such systems are viable and may provide advantages, without any imposed volume, area, or mass constraints.

Analytical Derivation

The problem to be considered is a fluid is to be delivered a distance x away from the inlet, at a given mass flow rate in a branching system shown in Fig. 1 for a 4 level bifurcating system.

The analysis is carried out for laminar, fully developed flow neglecting minor losses.

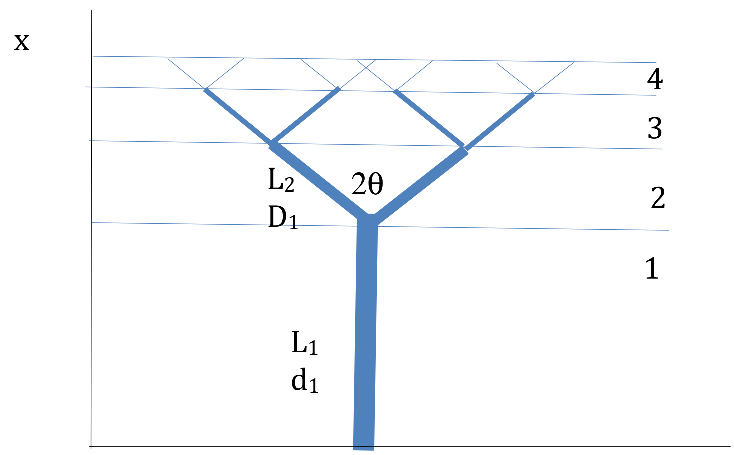

Figure 1: 2D Geometry of the Branching system

The pressure drop for laminar flow in a circular parallel pipe of diameter d that is to deliver a total mass flow rate of of $\dot{m}$ a horizontal distance x is given by

$$\Delta P = G\frac{\nu \dot{m} x}{d_h^4} \tag{4}$$

where G is

$$G = \begin{cases} \dfrac{2C}{\pi} & \text{circular pipe} \\ \dfrac{2C}{\dfrac{4(1+\alpha)^2}{\alpha}} & \text{rectangular pipe} \end{cases} \tag{5}$$

where C is the Poiseuille constant. For a circular pipe C=64, and for a rectangular pipe [7]

$$C = 96\left(1 - 1.3553\alpha + 1.9467\alpha^2 - 1.7012\alpha^3 + 0.9564\alpha^4 - 0.2537\alpha^5\right) \tag{6}$$

where $\alpha < 1$ is the aspect ratio

For a branching system, having a parent and n-1 children for a total of n generations, the total flow length is the sum of the lengths of the individual branched pipes, where x represents the straight distance from the inlet of the first pipe to the outlet of the last, and the branching angle, θ, is the half angle between the branched pipes, having a maximum of 45°. In terms of the branch lengths, x is given by

$$x = L_1 + \left(L_2 + ...L_n\right)\cos\theta \tag{7}$$

If $\lambda > 1$ is the parent-to-child length ratio

$$\lambda = \frac{L_i}{L_{i+1}} \quad (8)$$

Then

$$L_i = \frac{L_1}{\lambda^{i-1}} \quad (9)$$

and Eq. (7) can be solved for the length of the first (parent) branch

$$L_1 = \frac{x}{(1-\cos\theta) + \dfrac{\dfrac{1}{\lambda^n}-1}{\dfrac{1}{\lambda}-1}\cos\theta} \quad (10)$$

Similarly, we can define the diameter ratio

$$\delta = \frac{d_i}{d_{i+1}} \quad (11)$$

and therfore,

$$d_i = \frac{d_1}{\delta^{i-1}} \quad (12)$$

The mass flow rate at each branch level i is

$$\dot{m}_i = \frac{\dot{m}_1}{c^{i-1}} \quad (13)$$

where c is the number of children in each generation, c, assumed constant in all generations. The total pressure drop is given by

$$\Delta P_b = \sum \Delta P_i \quad (14)$$

For fully developed flow in a circular pipe

$$\Delta P_i = \frac{128\nu}{\pi}\dot{m}_i \frac{L_i}{d_i^{\,4}} = \frac{128\nu}{\pi}\frac{\dot{m}_1 L_1}{d_1}\left(\frac{\delta^4}{c\lambda}\right)^{i-1} \quad (15)$$

which when substituted in Eq. (14) and simplified results in an explicit expression for pressure drop in a branching system with n generations.

$$\Delta P_b = \frac{128\nu\dot{m}L_1}{\pi d_1^{\,4}} \frac{\left(\dfrac{\delta^4}{c\lambda}\right)^n - 1}{\left(\dfrac{\delta^4}{c\lambda}\right) - 1} \quad (16)$$

Note that if the distance x is known, L_1 can be determined from Eq. (10).

This equation reveals something interesting, in that the parameter $\frac{\delta^4}{c\lambda}$ must be less than one

$$\frac{\delta^4}{c\lambda} < 1 \tag{17}$$

otherwise the pressure drop increases exponentially, with increasing number of generations.

The total area of the branching system is

$$A_b = \sum_1^{n-1} A_i = \sum_1^{n-1} c^{i-1} \pi d_i L_i \tag{18}$$

which results in

$$A_b = \pi d_1 L_1 \frac{\left(\frac{c}{\delta\lambda}\right)^n - 1}{\left(\frac{c}{\delta\lambda}\right) - 1} \tag{19}$$

To avoid area increasing exponentially requires

$$\frac{c}{\delta\lambda} < 1 \tag{20}$$

The mass of the branching system is calculated by adding the mass at each generation

$$m_b = \rho_s \sum \frac{\pi}{4}\left[\left(d_i + \tau d_i\right)^2 - d_i^2\right] L_i c^{i-1} = \rho_s\left(\tau^2 + 2\tau\right)\frac{\pi}{4}\sum d_i^2 L_i c^{i-1} \tag{21}$$

which simplifies to

$$m_b = \rho_s\left(\tau^2 + 2\tau\right)\frac{\pi}{4} d_1^2 L_1 \frac{\left(\frac{c}{\lambda\delta^2}\right)^n - 1}{\left(\frac{c}{\lambda\delta^2}\right) - 1} \tag{22}$$

where it is assumed that the thickness of the pipe, t, is proportional to pipe inner diameter,

$$t = \tau d \tag{23}$$

This condition results in constant hoop stress in all the branches and ensures structural soundness of the system, that if the first branch can withstand the system pressure, the last branch, which will be at a lower pressure and smaller diameter will also be able to.

Again, to have a finite mass

$$\frac{c}{\lambda\delta^2} < 1 \tag{24}$$

Results

The closed form solution provided by Eq. (20) provides much insight into the behavior of the branching systems including fractal-like branching micro-channel heat sinks. It is based on the assumption that flow is laminar and fully developed. The fully developed assumption is an approximation. On the one hand, it under-predicts pressure drop, since it does not account for flow acceleration at the inlet, and on the other hand it over-predicts since it does not account for pressure recovery at the branching because of area expansion. A number of other studies are available and we compare their results with the predictions of the model.

Table 1 compares the pressure drop predictions of the model with those of some available results, and as can be seen the results are in reasonable agreement, specially since the available results are for rectangular branching systems and Eq. (16) is for a circular one.

Table 1 Comparison with available results

Ref.	n	Hk(mm)	wk(mm)	dhk (m	Lk(mm)	δ	λ	c	Vol	ΔP ref	ΔP, curr	Error
[8]	4	0.5	1.285	0.72	12.45	1.26	1.41	2	0.65	0.5	0.6	13%
[8]	4	0.5	1.285	0.72	12.45	1.26	1.413	2	5	4.2	4.6	10%
[8]	4	0.5	1.285	0.72	12.45	1.26	1.413	2	15	12.9	13.8	7%
[8]	4	0.5	1.285	0.72	12.45	1.26	1.413	2	25	22.2	23.1	4%
[9]	4	0.25	0.539	0.342	5.8	1.26	1.415	2	1.8	88.0	81.3	8%
[10]	4	0.25	0.643	0.36	6.23	1.259	1.416	2	0.90	83.0	78.0	6%

Alharbi [9] also conducted a 3D numerical investigation and their pressure drop results from the CFD analyses is 50% lower than the one dimensional results presented in Table 1.

In order to determine the accuracy of Eq. (14), we also conducted a CFD analysis. The geometry shown in Fig. 2 was generated in Creo Parametric and imported into Ansys for meshing and analysis.

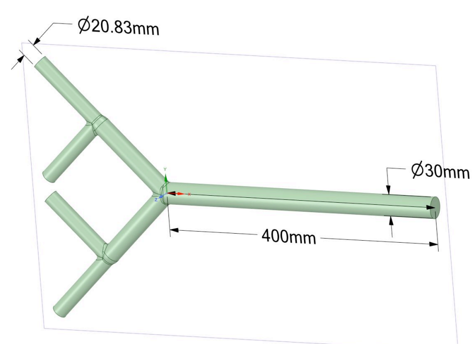

Figure 2: Branching system SpaceClaim model

To ensure fully developed flow at least in the parent and the final generation, extension pipes were added to the inlet of the first generation and the outlets of the last. A 200 mm extension was added to the first generation of the first generation, and the exit 4^{th} generation was extended by an additional 25 mm. The outlet of the 25 mm extension was specified as pressure outlet with a magnitude of zero.

Because the branching shape is mostly cylindrical (curved) and contains small protrusions at the bifurcation regions, tetrahedron unstructured meshing was implemented. Tetrahedron mesh, as opposed to a quadrilateral mesh, more effectively discretizes curved surfaces. Additionally, to ensure a smooth transition between the mesh refinement on the pipe's surface and the inner fluid domain, an inflation of 10 layers with a growth factor of 1.2 was implemented. The inflation layers at the interior surface of the pipe captures the changes in the boundary layer more accurately. Face sizing and edge sizing has also been used to create the mesh. A total of 8,619,594 elements and 3,044,681 nodes were used.

The non-dimensional x- component of the centerline velocity of the 1st generation of the branching system and that of a single pipe (parallel pipe) is plotted against the non-dimensional distance along pipe in Fig. 3.

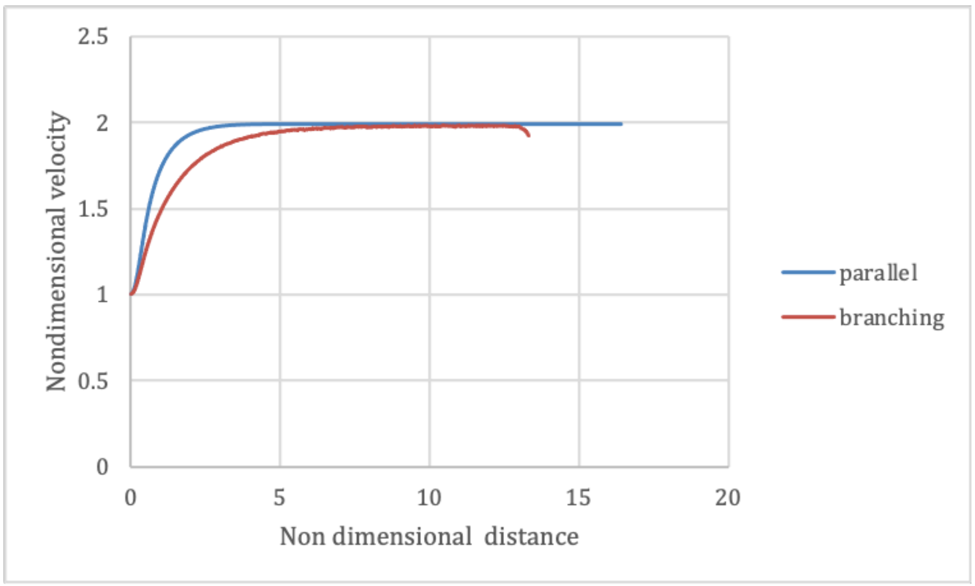

Figure 3: Non-dimensional velocity vs. non-dimensional distance

The dimensionless velocities should reach a value of 2, twice the centerline inlet velocity, when the flow becomes fully developed. The pressure drop was calculated by subtracting the pressure at 25 mm the outlet of the branching system (before the extension section) from the pressure at x=200 mm (inlet of the branching system after the extension section). The CFD results provided a pressure drop of

$$\Delta P = 1.2803 \times 10^{-2}$$

in the branching system. For the conditions given, Eq. (14) predicts a pressure drop of $\Delta P = 1.3803 \times 10^{-2}$, which is different by 8%.

The pressure for the entire system was exported from Ansys and compared to the analytical results using excel. The exact analytical results were tabulated in excel and plotted with the data retrieved from Ansys and are shown in Fig. 4.

As can be seen from Fig. 4, there are slight differences between the branches because of small differences in the branch mass flow rate. Since the mass flow rate is not exactly divided by two at the bifurcation region. The CFD simulations also reached fully developed conditions in the first and 4th generations as a result of adding the extensions. The 3 dimensional CFD simulation results for pressure drop is in close agreement with the analytical results based on a fully developed flow throughout the whole system.

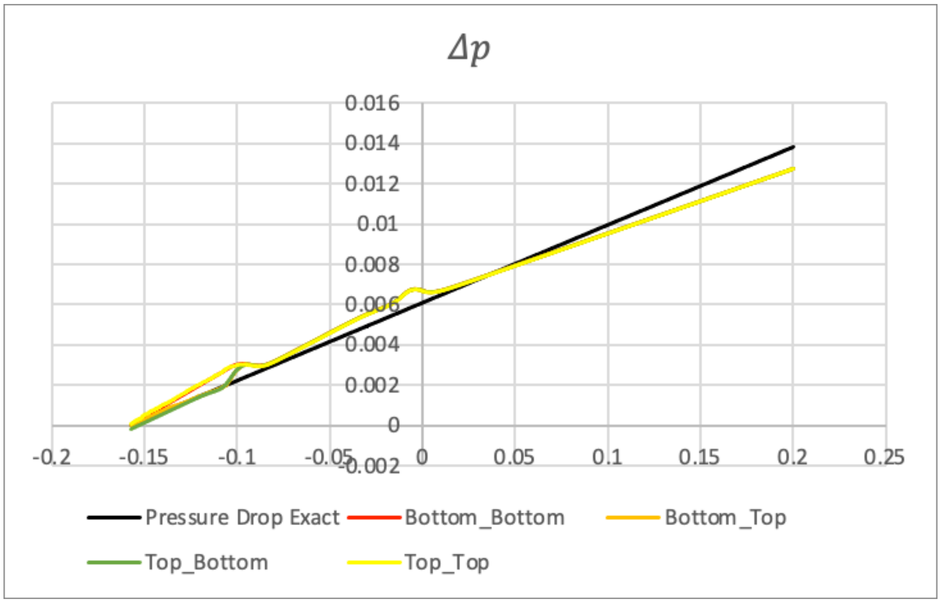

Figure 4: Pressure drop comparison of Numerical and the analytical Predictions

As mentioned above, for a branching system to have finite mass, area, and pressure drop, three criteria different have to be met. The criterion for finite surface is a stronger than the one for finite mass, i.e. if Eq. (20) is satisfied, so will Eq. (24). The remaining two criteria result in

$$\frac{\delta^4}{\lambda} < c < \delta\lambda \tag{25}$$

This is an interesting conclusion which sets the criteria for a branching system to be a physically viable option by being able to grow and exist at different scales. In a sense, it is setting the condition for it to function as a fractal system and continue to replicate itself over a broad range of scales and remain viable.

From Murray's law, for a bifurcating system

$$\delta = 2^{1/3} \tag{26}$$

Substituting in Eq. (25), imposes a minimum for λ in a bifurcating system, requiring that

$$\lambda > 2^{2/3} \tag{27}$$

Meeting this condition also satisfies the other requirement that

$$\frac{\delta^4}{\lambda} = \frac{2^{4/3}}{2^{2/3}} = 2^{2/3} < 2\,.$$

These results of Eq. (25) are interesting and different than some of what has been proposed. Bejan and Laurent [11] argue that when channels bifurcate or coalesce their diameters should change so that the overall flow through the architecture is facilitated, and indicate that happens for $\delta = 2^{1/3}$ and $\lambda = 2^{1/3}$. These values result in $\frac{\delta^4}{c\lambda} = 1$ and $\frac{c}{\delta\lambda} = 1.26$ and are expected to lead to increasing pressure drop, area, and mass in the branching system with increasing number of generations, which will not make such a system physically viable if they are expected to grow.

This is illustrated by an example. Consider water at the rate of 0.314 kg/s is to be delivered over a distance of 1000 m in a branching system whose initial pipe diameter is 0.4 m. All flow specifications are shown in Table 2 for two cases, one satisfying the constraints (25) and another following recommendations of Constructal law.

Table 2 Flow Conditions

x (m)	1000.00	1000.00
d_1 (m)	0.4	0.4
c	2.00	2.00
n	20.00	20.00
θ (rad)	0.3927	0.3927
ν (m^2/s)	1.05E-06	1.05E-06
ρ (kg/m^3)	1.00E+03	1.00E+03
ρ_s (kg/m^3)	8.96E+03	8.96E+03
μ (Pa.s)	1.05E-03	1.05E-03
V (m/s)	2.50E-03	2.50E-03
δ	1.31	1.26
λ	1.70	1.26
τ	3%	3%
mass flow (kg/s)	3.14E-01	3.14E-01
Re_1	950.75	950.75
$\delta^4/c\lambda$	0.86	1.00
$c/\delta\lambda$	0.90	1.26
$c/\delta^2\lambda$	0.69	1.00

As can be seen in Fig. 5, the constraints lead to pressure drop and the system mass to asymptote to constant values. For this case, the driving potential (pressure drop) for the flow remains constant as the number of generations increases, or pressure drop does not prevent the number of generations to increase. If the constrains are not met, the pressure drop and mass of the branching system grow exponentially (in this case linearly since $\frac{\delta^4}{c\lambda} = 1$) and the system becomes impractical. For the example, the pressure drop is 1.5 times and the mass is over three times more after 20 generations.

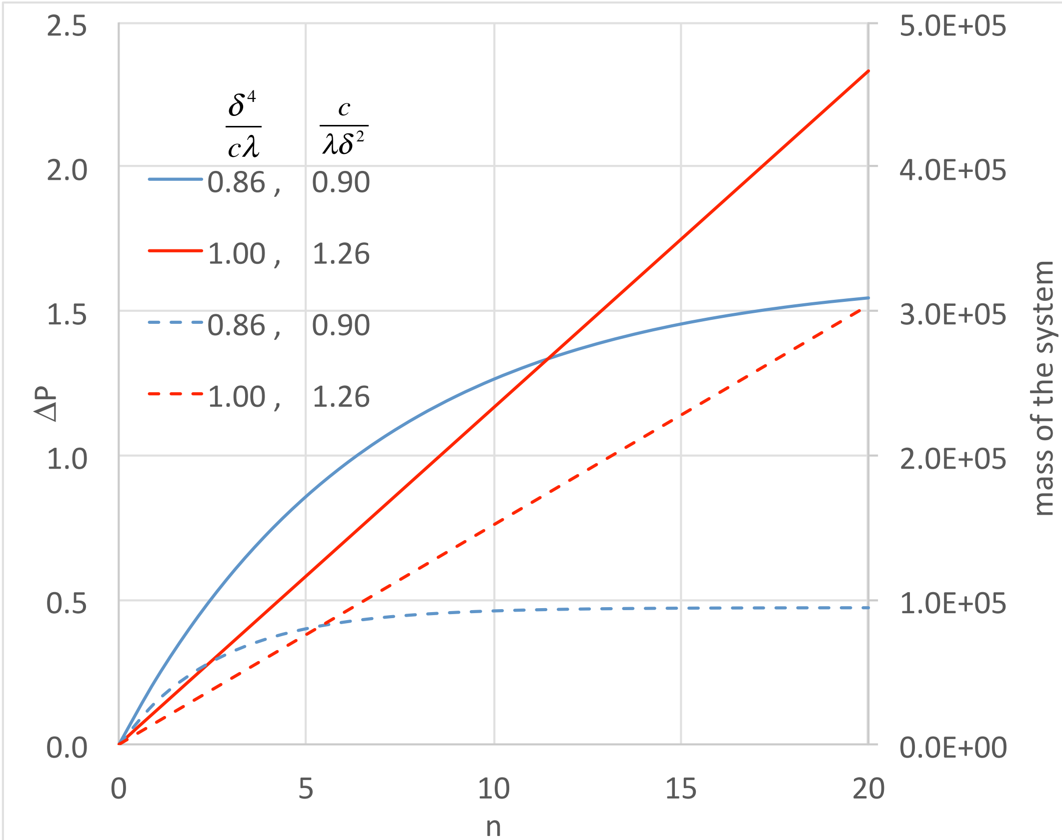

Figure 5 Variation of pressure drop (solid lines) and mass (dashed lines) with generations

Conclusion

The analytical model developed is able to predict the pressure drop in branching systems and is used to develop the criteria for them to be physically viable and replicating. Essentially, the model states that for branching systems to be viable $\frac{\delta^4}{\lambda} < c < \delta\lambda$ which are the criteria for these systems to be fractal.

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